

Both necessary and sufficient conditions for Bayesian exponential consistency

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Abstract

The last decade has seen a remarkable development in the theory of asymptotics of Bayesian nonparametric procedures. Exponential consistency has played an important role in this area. It is known that the condition of f_0 being in the Kullback-Leibler support of the prior cannot ensure exponential consistency of posteriors. Many authors have obtained additional sufficient conditions for exponential consistency of posteriors, see, for instance, Schwartz [7], Barron, Schervish and Wasserman [2], Ghosal, Ghosh and Ramamoorthi [5], Walker [8], Xing and Ranneby [12]. However, given the Kullback-Leibler support condition, less is known about both necessary and sufficient conditions. In this paper we give one type of both necessary and sufficient conditions. As a consequence we derive a simple sufficient condition on Bayesian exponential consistency, which is weaker than the previous sufficient conditions.

Keywords: Bayesian consistency, prior distribution, infinite-dimensional model.

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1 Introduction

Let X_1, X_2, \dots, X_n be an independent identically distributed sample of n random variables, taking values in a Polish space \mathbb{X} endowed with a σ -algebra \mathcal{X} and having a common density f with respect to a dominated σ -finite measure μ on \mathbb{X} . For any two densities f and g , the Hellinger distance is $H(f, g) = \left(\int_{\mathbb{X}} (\sqrt{f(x)} - \sqrt{g(x)})^2 \mu(dx) \right)^{1/2}$ and the Kullback-Leibler divergence is $K(f, g) = \int_{\mathbb{X}} f(x) \log \frac{f(x)}{g(x)} \mu(dx)$. Assume that the space \mathbb{F} of densities is separable with respect to the Hellinger metric and assume that \mathcal{F} is the Borel σ -algebra of \mathbb{F} . For a prior Π on \mathbb{F} , the posterior is the conditional distribution of Π , given X_1, X_2, \dots, X_n , with the following expression

$$\Pi(A \mid X_1, X_2, \dots, X_n) = \frac{\int_A \prod_{i=1}^n f(X_i) \Pi(df)}{\int_{\mathbb{F}} \prod_{i=1}^n f(X_i) \Pi(df)} = \frac{\int_A R_n(f) \Pi(df)}{\int_{\mathbb{F}} R_n(f) \Pi(df)}$$

for measurable subsets $A \subset \mathbb{F}$, where $R_n(f) = \prod_{i=1}^n \{f(X_i)/f_0(X_i)\}$ stands for the likelihood ratio. If the posterior $\Pi(\cdot \mid X_1, X_2, \dots, X_n)$ concentrates on arbitrarily small neighborhoods of the true density f_0 almost surely or in probability, then it is said to be consistent at f_0 almost surely or in probability respectively, where the almost sure convergence and the convergence in probability are with respect to the infinite product distribution $P_{f_0}^\infty$ of the true distribution P_{f_0} with the density f_0 . The true density f_0 is said to be in the Kullback-Leibler support of the prior Π if $\Pi(f : K(f_0, f) < \varepsilon) > 0$ for each $\varepsilon > 0$.

Consistency plays an important role in statistics. Early works on Bayesian nonparametric procedures were concerned with weak consistency of posteriors. Freedman [4] and Diaconis and Freedman [3] proved that a prior with positive mass on each weak neighborhood of f_0 cannot imply the weak consistency of posteriors. A sufficient condition on weak consistency was provided by Schwartz [7], who proved that if f_0 is in the Kullback-Leibler support of Π then the posteriors accumulate in all weak neighborhoods of f_0 . However, the Kullback-Leibler support condition is not enough to guarantee almost sure consistency of posteriors. Assume now that f_0 is in the Kullback-Leibler support of Π . Barron, Schervish and Wasserman [2], Ghosal, Ghosh and Ramamoorthi [5], Walker [8], Xing and Ranneby [12] have obtained some sufficient conditions for posteriors to be almost surely consistent. The approaches of Barron et al. [2] and Ghosal et al. [5] are to construct suitable sieves and to compute

metric entropies. Their works were discussed in great detail in the monograph of Ghosh and Ramamoorthi [6], see also the nice review of Wasserman [9]. Walker's result [8] relies upon summability of squareroots of prior probability of suitable coverings. Xing and Ranneby [12] used the Hausdorff α -entropy to deal with the problem. In fact, all these almost sure consistency results are to establish sufficient conditions on exponential consistency of posteriors, i.e., posterior probabilities exponentially tend to zero. Much less is known about both necessary and sufficient conditions for exponential consistency of posteriors. To our knowledge there only exists a both necessary and sufficient condition due to Barron [1], who used uniformly consistent tests to describe exponential consistency of posteriors. Barron's result has been widely applied in practice. In this paper we provide one type of both necessary and sufficient conditions for exponential consistency of posteriors. Our result shall be applied to give a verification of Barron's condition. As a consequence of our results we obtain a sufficient condition for exponential consistency of posteriors, which implies several well known sufficient conditions.

2 Consistency of Posterior Distributions

In the section we give both necessary and sufficient conditions in the two senses: almost sure and in-probability. Some applications and consequences are discussed.

We consider \mathbb{X}^n as a subset of \mathbb{X}^∞ by identifying $(x_1, x_2, \dots, x_n) \in \mathbb{X}^n$ with the point $(x_1, x_2, \dots, x_n, 0, 0, \dots) \in \mathbb{X}^\infty$. For a sequence $\{D_n\}_1^\infty$ of subsets $D_n \subset \mathbb{X}^n$, we denote

$$\limsup D_n = \{(x_1, x_2, \dots) \in \mathbb{X}^\infty : (x_1, x_2, \dots, x_n) \in D_n \text{ infinitely often}\}.$$

Barron [1] investigated relationship between exponential posterior consistency and existence of uniformly consistent tests. He obtained a characterization of exponential posterior consistency. Now we give one new type of characterizations of exponential posterior consistency.

Theorem 1. *Suppose that the true density f_0 is in the Kullback-Leibler support of Π and that $\{A_n\}_1^\infty$ be a sequence of subsets of \mathbb{F} . Then the following statements are equivalent.*

- (i) *There exists a constant $\beta_0 > 0$ such that*

$$e^{n\beta_0} \Pi(A_n | X_1, X_2, \dots, X_n) \longrightarrow 0$$

almost surely as $n \rightarrow \infty$.

(ii) There exists a constant $\beta_1 > 0$ such that

$$P_{f_0}^\infty \left\{ \Pi(A_n \mid X_1, X_2, \dots, X_n) > e^{-n\beta_1} \text{ infinitely often} \right\} = 0.$$

(iii) There exist constants $0 < \alpha_1 \leq 1$, $\beta_2 > 0$ and a sequence $\{D_n\}_1^\infty$ of sets $D_n \subset \mathbb{X}^n$ with $P_{f_0}^\infty(\limsup D_n) = 0$ such that

$$E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) \right)^{\alpha_1} \leq e^{-n\beta_2} \quad \text{for all large } n,$$

where E_{f_0} stands for the expectation with respect to X_1, X_2, \dots, X_n and $1_{\mathbb{X}^n \setminus D_n}$ denotes the indicator function of $\mathbb{X}^n \setminus D_n$.

(iv) For each $0 < \alpha \leq 1$ there exist a constant $\beta_\alpha > 0$ and a sequence $\{D_n\}_1^\infty$ of sets $D_n \subset \mathbb{X}^n$ with $P_{f_0}^\infty(\limsup D_n) = 0$ such that

$$E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) \right)^\alpha \leq e^{-n\beta_\alpha} \quad \text{for all large } n.$$

(v) There exist a constant $\beta_3 > 0$ and a sequence $\{D_n\}_1^\infty$ of sets $D_n \subset \mathbb{X}^n$ such that $P_{f_0}^\infty(\limsup D_n) = 0$ and

$$\int_{A_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) \leq e^{-n\beta_3} \quad \text{for all large } n.$$

Note that for reader's convenience we include (ii) in Theorem 1 even though that the equivalence of (i) and (ii) is clear.

Proof of Theorem 1. The implications (i) \Leftrightarrow (ii) and (iv) \Rightarrow (iii) are trivial. The equivalence (iii) \Leftrightarrow (v) follows directly from the equality

$$\begin{aligned} E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) \right) &= \int_{A_n} E_{f_0} (1_{\mathbb{X}^n \setminus D_n} R_n(f)) \Pi(df) \\ &= \int_{A_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df). \end{aligned}$$

So it suffices to prove (ii) \Rightarrow (iv) and (iii) \Rightarrow (ii). To prove (ii) \Rightarrow (iv), we set

$$D_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{X}^n : \Pi(A_n \mid x_1, x_2, \dots, x_n) > e^{-n\beta_1}\}.$$

Then by (ii) we have $P_{f_0}^\infty(\limsup D_n) = 0$. Write

$$A_n = \{f \in A_n : P_f^\infty(\mathbb{X}^n \setminus D_n) \geq e^{-\frac{n\beta_1}{2}}\} \cup \{f \in A_n : P_f^\infty(\mathbb{X}^n \setminus D_n) < e^{-\frac{n\beta_1}{2}}\}$$

$$:= A_n^1 \cup A_n^2.$$

Given $0 < \alpha \leq 1$, by the inequality $(s+t)^\alpha \leq s^\alpha + t^\alpha$ for $s, t \geq 0$ we have

$$\begin{aligned} & E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) \right)^\alpha \\ & \leq E_{f_0} \left(\int_{A_n^1} R_n(f) \Pi(df) \right)^\alpha + E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n^2} R_n(f) \Pi(df) \right)^\alpha. \end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned} & E_{f_0} \left(\int_{A_n^1} R_n(f) \Pi(df) \right)^\alpha \leq \left(E_{f_0} \left(\int_{A_n^1} R_n(f) \Pi(df) \right)^{\alpha \cdot \frac{1}{\alpha}} \right)^\alpha \left(E_{f_0} 1^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \\ & = \left(\int_{A_n^1} (E_{f_0} R_n(f)) \Pi(df) \right)^\alpha = \left(\int_{A_n^1} \Pi(df) \right)^\alpha \leq e^{\frac{n\beta_1\alpha}{2}} \left(\int_{A_n^1} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) \right)^\alpha \\ & = e^{\frac{n\beta_1\alpha}{2}} \left(\int_{\mathbb{X}^n \setminus D_n} \left(\int_{A_n^1} \prod_{i=1}^n f(x_i) \Pi(df) \right) \mu(dx_1) \dots \mu(dx_n) \right)^\alpha \\ & = e^{\frac{n\beta_1\alpha}{2}} \left(\int_{\mathbb{X}^n \setminus D_n} \left(\int_{A_n^1} R_n(f) \Pi(df) \right) \prod_{i=1}^n f_0(x_i) \mu(dx_1) \dots \mu(dx_n) \right)^\alpha, \end{aligned}$$

which by the definition of D_n does not exceed

$$\begin{aligned} & e^{\frac{n\beta_1\alpha}{2}} e^{-n\beta_1\alpha} \left(\int_{\mathbb{X}^n \setminus D_n} \left(\int_{\mathbb{F}} R_n(f) \Pi(df) \right) \prod_{i=1}^n f_0(x_i) \mu(dx_1) \dots \mu(dx_n) \right)^\alpha \\ & = e^{-\frac{n\beta_1\alpha}{2}} \left(\int_{\mathbb{F}} \left(\int_{\mathbb{X}^n \setminus D_n} \prod_{i=1}^n f(x_i) \mu(dx_1) \dots \mu(dx_n) \right) \Pi(df) \right)^\alpha \leq e^{-\frac{n\beta_1\alpha}{2}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n^2} R_n(f) \Pi(df) \right)^\alpha \leq \left(E_{f_0} \int_{A_n^2} 1_{\mathbb{X}^n \setminus D_n} R_n(f) \Pi(df) \right)^\alpha \\ & = \left(\int_{A_n^2} E_{f_0} (1_{\mathbb{X}^n \setminus D_n} R_n(f)) \Pi(df) \right)^\alpha = \left(\int_{A_n^2} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) \right)^\alpha \\ & \leq \left(\int_{A_n^2} e^{-\frac{n\beta_1}{2}} \Pi(df) \right)^\alpha \leq e^{-\frac{n\beta_1\alpha}{2}}. \end{aligned}$$

Thus, we have obtained (iv) for $\beta_\alpha = \frac{\beta_1\alpha}{4}$.

Now we prove (iii) \Rightarrow (ii). Note that

$$\begin{aligned} & \Pi(A_n | X_1, X_2, \dots, X_n) \\ &= 1_{D_n} \Pi(A_n | X_1, X_2, \dots, X_n) + 1_{\mathbb{X}^n \setminus D_n} \Pi(A_n | X_1, X_2, \dots, X_n) \end{aligned}$$

and

$$P_{f_0}^\infty(1_{D_n} \neq 0 \text{ infinitely often}) = P_{f_0}^\infty(\limsup D_n) = 0.$$

So for $\beta_1 = \frac{\beta_2}{3\alpha_1}$ with the constants β_2 and α_1 from (iii), we have

$$\begin{aligned} & P_{f_0}^\infty\{\Pi(A_n | X_1, X_2, \dots, X_n) > e^{-n\beta_1} \text{ infinitely often}\} \\ &= P_{f_0}^\infty\{1_{\mathbb{X}^n \setminus D_n} \Pi(A_n | X_1, X_2, \dots, X_n) > e^{-n\beta_1} \text{ infinitely often}\} \\ &\leq P_{f_0}^\infty\left\{1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) > e^{-n2\beta_1} \text{ infinitely often}\right\}, \end{aligned}$$

where the last inequality follows from

$$P_{f_0}^\infty\left\{\int_{\mathbb{F}} R_n(f) \Pi(df) \leq e^{-n\beta_1} \text{ infinitely often}\right\} = 0,$$

see Lemma 4 of Barron et al. [2]. On the other hand, by (iii) we have

$$\begin{aligned} & P_{f_0}^\infty\left\{1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) > e^{-n2\beta_1}\right\} \\ &\leq P_{f_0}^\infty\left\{\left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df)\right)^{\alpha_1} > e^{-n2\beta_1\alpha_1}\right\} \\ &\leq e^{n2\beta_1\alpha_1} E_{f_0}\left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df)\right)^{\alpha_1} \leq e^{n2\beta_1\alpha_1 - n\beta_2} = e^{-n\beta_2/3}, \end{aligned}$$

which by the first Borel-Cantelli Lemma yields that

$$P_{f_0}^\infty\left\{1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) > e^{-n2\beta_1} \text{ infinitely often}\right\} = 0.$$

Thus we have obtain (ii) and the proof of Theorem 1 is complete. \square

As an application of Theorem 1 we prove the following characterization of Barron [1], see also Theorem 4.4.3 in Ghosh and Ramamoorthi [6]. Recall that a test is a measurable function ϕ satisfying $0 \leq \phi \leq 1$.

Corollary 1. (Barron [1]). Suppose that the true density f_0 is in the Kullback-Leibler support of Π and that $\{A_n\}_1^\infty$ be a sequence of subsets in \mathbb{F} . Then the following statements are equivalent.

- (i) There exists a constant $\beta_0 > 0$ such that

$$e^{n\beta_0} \Pi(A_n | X_1, X_2, \dots, X_n) \longrightarrow 0$$

almost surely as $n \rightarrow \infty$.

- (ii) There exist subsets V_n, W_n of \mathbb{F} , positive constants $c_1, c_2, \beta_1, \beta_2$ and a sequence of tests $\{\phi_n = \phi_n(X_1, \dots, X_n)\}$ such that
 - (a) $A_n \subset V_n \cup W_n$;
 - (b) $\Pi(W_n) \leq c_1 e^{-n\beta_1}$;
 - (c) $P_{f_0}^\infty\{\phi_n > 0 \text{ infinitely often}\} = 0$ and $\inf_{f \in V_n} E_f \phi_n \geq 1 - c_2 e^{-n\beta_2}$.

Proof. We need to prove that (ii) of Corollary 1 is equivalent to (v) of Theorem 1. Assume that (ii) holds. Set $D_n = \{\phi_n > 0\}$. Then $P_{f_0}^\infty(\limsup D_n) = 0$ and

$$\begin{aligned} \int_{A_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) &\leq \int_{V_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) + \int_{W_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) \\ &\leq \int_{V_n} E_f 1_{\mathbb{X}^n \setminus D_n} \Pi(df) + \int_{W_n} \Pi(df) \leq \int_{V_n} E_f((1 - \phi_n) 1_{\mathbb{X}^n \setminus D_n}) \Pi(df) + c_1 e^{-n\beta_1} \\ &\leq \int_{V_n} c_2 e^{-n\beta_2} \Pi(df) + c_1 e^{-n\beta_1} \leq c_2 e^{-n\beta_2} + c_1 e^{-n\beta_1}, \end{aligned}$$

which implies (v) for $\beta_3 = (\beta_1 \wedge \beta_2)/2$. Conversely, assume that (v) holds. So for $\phi_n = 1_{D_n}$ we have that $P_{f_0}^\infty\{\phi_n > 0 \text{ infinitely often}\} = P_{f_0}^\infty(\limsup D_n) = 0$. Take $W_n = \{f \in A_n : P_f^\infty(\mathbb{X}^n \setminus D_n) \geq e^{-n\beta_3/2}\}$ and $V_n = A_n \setminus W_n$. Then

$$e^{-n\beta_3/2} \Pi(W_n) \leq \int_{A_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) \leq e^{-n\beta_3} \quad \text{for all large } n,$$

which implies (b) for $\beta_1 = \beta_3/2$, and for each $f \in V_n$ we have

$$1 - E_f \phi_n = 1 - P_f^\infty(D_n) = P_f^\infty(\mathbb{X}^\infty \setminus D_n) = P_f^\infty(\mathbb{X}^n \setminus D_n) \leq e^{-n\beta_3/2},$$

which yields (c). Hence we have obtained (ii) and the proof of Corollary 1 is complete. \square

Theorem 1 can be used to develop sufficient conditions for exponential posterior consistency.

Definition 1. Let d be a metric on \mathbb{F} . The posterior distribution $\Pi(\cdot | X_1, X_2, \dots, X_n)$ is said to be exponentially consistent at the true density f_0 almost surely (in probability) if for any $\varepsilon > 0$ there exists a constant $\beta_\varepsilon > 0$ such that

$$e^{n\beta_\varepsilon} \Pi(f : d(f, f_0) \geq \varepsilon | X_1, X_2, \dots, X_n) \longrightarrow 0$$

almost surely (in probability) as $n \rightarrow \infty$.

A direct consequence of Theorem 1 is the following result.

Corollary 2. Let d be a metric on \mathbb{F} and let r be a positive constant. Suppose that the true density f_0 is in the Kullback-Leibler support of Π and suppose that for any $\varepsilon > 0$ there exist constants $0 < \alpha_\varepsilon \leq 1$ and $\beta_\varepsilon > 0$ such that

$$E_{f_0} \left(\int_{\{f : d(f, f_0) \geq r\varepsilon\}} R_n(f) \Pi(df) \right)^{\alpha_\varepsilon} \leq e^{-n\beta_\varepsilon} \quad \text{for all large } n.$$

Then $\Pi(\cdot | X_1, X_2, \dots, X_n)$ is exponentially consistent at f_0 almost surely as $n \rightarrow \infty$.

Corollary 2 gives a sufficient condition for posterior consistency. It makes it possible to obtain posterior consistency without computation of metric entropies. In the following three corollaries we shall apply this sufficient condition to verify the conditions given by Ghosal, Ghosh and Ramamoorthi [5], Walker [8] and Xing and Ranneby [12].

Let L_μ be the space of all nonnegative integrable functions with the norm $\|f\| = \int_{\mathbb{X}} |f(x)| \mu(dx)$. Recall that the L_μ -metric entropy $J(\delta, \mathcal{G})$ is the logarithm of the minimum of all numbers N such that there exist f_1, f_2, \dots, f_N in L_μ satisfying $\mathcal{G} \subset \bigcup_{i=1}^N \{f \in L_\mu : \|f - f_i\| < \delta\}$, see Ghosal et al [5].

Corollary 3. (Ghosal et al [5]). Suppose that the true density function f_0 is in the Kullback-Leibler support of Π and suppose that for any $\varepsilon > 0$ there exist $0 < \delta < \frac{\varepsilon}{4}$, $c_1, c_2 > 0$, $0 < \beta < \frac{\varepsilon^2}{8}$, and $\mathcal{G}_n \subset \mathbb{F}$ such that for all large n ,

- (a) $\Pi(\mathbb{F} \setminus \mathcal{G}_n) < c_1 e^{-n c_2}$;
- (b) $J(\delta, \mathcal{G}_n) < n \beta$.

Then for any $\varepsilon > 0$,

$$\Pi(f : \|f - f_0\| \geq \varepsilon | X_1, X_2, \dots, X_n) \longrightarrow 0$$

almost surely as $n \rightarrow \infty$.

Proof. Given $\varepsilon > 0$, by the inequality $(s + t)^{1/2} \leq s^{1/2} + t^{1/2}$ for $s, t \geq 0$ we have

$$\begin{aligned} & E_{f_0} \left(\int_{\{f: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2} \\ & \leq E_{f_0} \left(\int_{\mathbb{F} \setminus \mathcal{G}_n} R_n(f) \Pi(df) \right)^{1/2} + E_{f_0} \left(\int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2}, \end{aligned}$$

which, by Hölder's inequality and (b), we have

$$\begin{aligned} & E_{f_0} \left(\int_{\mathbb{F} \setminus \mathcal{G}_n} R_n(f) \Pi(df) \right)^{1/2} \leq \left(E_{f_0} \int_{\mathbb{F} \setminus \mathcal{G}_n} R_n(f) \Pi(df) \right)^{1/2} \\ & = \left(\int_{\mathbb{F} \setminus \mathcal{G}_n} E_{f_0} R_n(f) \Pi(df) \right)^{1/2} = \Pi(\mathbb{F} \setminus \mathcal{G}_n)^{1/2} < c_1 e^{-n c_2 / 2}. \end{aligned}$$

On the other hand, by the proof of Theorem 2 in Ghosal et al [5], we know that (b) implies that there exist tests ϕ_n such that

$$E_{f_0} \phi_n \leq e^{-n(\varepsilon^2/8-\beta)} \quad \text{and} \quad \inf_{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon} E_f \phi_n \geq 1 - e^{-2n(\varepsilon/4-\delta)^2}.$$

Hence by Hölder's inequality we get

$$\begin{aligned} & E_{f_0} \left(\int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2} \\ & \leq E_{f_0} \left(\phi_n \int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2} \\ & \quad + E_{f_0} \left((1 - \phi_n) \int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2} \\ & \leq (E_{f_0} \phi_n)^{1/2} \left(E_{f_0} \int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2} \\ & \quad + \left(E_{f_0} \left((1 - \phi_n) \int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right) \right)^{1/2} \\ & \leq e^{-n(\varepsilon^2/16-\beta/2)} \left(\int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} \Pi(df) \right)^{1/2} \\ & \quad + \left(\int_{\{f \in \mathcal{G}_n: \|f - f_0\| \geq \varepsilon\}} E_f (1 - \phi_n) \Pi(df) \right)^{1/2} \leq e^{-n(\varepsilon^2/16-\beta/2)} + e^{-n(\varepsilon/4-\delta)^2}. \end{aligned}$$

Thus, using Corollary 2, we have proved Corollary 3. \square

Corollary 4. (Walker [8]). Suppose that the true density f_0 is in the Kullback-Leibler support of Π and suppose that for any $\varepsilon > 0$ there exist a covering $\{A_1, A_2, \dots, A_j, \dots\}$ of $\{f : H(f, f_0) \geq \varepsilon\}$ and $0 < \delta < \varepsilon$ such that $\sum_{j=1}^{\infty} \sqrt{\Pi(A_j)} < \infty$ and each $A_j \subset \{f : H(f_j, f) < \delta\}$ for some density f_j satisfying $H(f_j, f_0) > \varepsilon$. Then for any $\varepsilon > 0$,

$$\Pi(f : H(f, f_0) \geq \varepsilon \mid X_1, X_2, \dots, X_n) \longrightarrow 0$$

almost surely as $n \rightarrow \infty$.

Proof. Denote $f_{kA_j}(x) = \int_{A_j} f(x) R_k(f) \Pi_n(df) / \int_{A_j} R_k(f) \Pi_n(df)$ and $R_0(f) = 1$. Write

$$\int_{A_j} R_n(f) \Pi_n(df) = \Pi_n(A_j) \prod_{k=0}^{n-1} \frac{f_{kA_j}(X_{k+1})}{f_0(X_{k+1})}.$$

Then for any $\varepsilon > 0$, by the inequality $(s+t)^{1/2} \leq s^{1/2} + t^{1/2}$ for $s, t \geq 0$, we have

$$\begin{aligned} E_{f_0} \left(\int_{\{f : \|f - f_0\| \geq \varepsilon\}} R_n(f) \Pi(df) \right)^{1/2} &\leq \sum_{j=1}^{\infty} E_{f_0} \left(\int_{A_j} R_n(f) \Pi(df) \right)^{1/2} \\ &= \sum_{j=1}^{\infty} \sqrt{\Pi_n(A_j)} E_{f_0} \left(\prod_{k=0}^{n-1} \frac{f_{kA_j}(X_{k+1})}{f_0(X_{k+1})} \right)^{1/2}. \end{aligned}$$

Hence, by Corollary 2, it is enough to show that there exists $\beta > 0$ such that for all j and n ,

$$E_{f_0} \left(\prod_{k=0}^{n-1} \frac{f_{kA_j}(X_{k+1})}{f_0(X_{k+1})} \right)^{1/2} \leq e^{-n\beta}.$$

Using Jensen's inequality we have $H(f_{kA_j}, f_j)^2 \leq \delta^2$ and hence $H(f_{kA_j}, f_0) \geq H(f_j, f_0) - H(f_j, f_{kA_j}) \geq \varepsilon - \delta > 0$. It then follows from Fubini's theorem that the last expectation is equal to

$$\begin{aligned} &\int_{\mathbb{X}^{n-1}} \int_{\mathbb{X}} \sqrt{f_{n-1A_j}(x_n) f_0(x_n)} \mu(dx_n) \prod_{k=0}^{n-2} \sqrt{f_{kA_j}(x_{k+1}) f_0(x_{k+1})} \mu(dx_1) \dots \mu(dx_{n-1}) \\ &= \int_{\mathbb{X}^{n-1}} \left(1 - \frac{H(f_{n-1A_j}, f_0)^2}{2} \right) \prod_{k=0}^{n-2} \sqrt{f_{kA_j}(x_{k+1}) f_0(x_{k+1})} \mu(dx_1) \dots \mu(dx_{n-1}) \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 - \frac{(\varepsilon - \delta)^2}{2}\right) \int_{\mathbb{X}^{n-1}} \prod_{k=0}^{n-2} \sqrt{f_{kA_j}(x_{k+1})f_0(x_{k+1})} \mu(dx_1) \dots \mu(dx_{n-1}) \\
&\leq \dots \text{using the same argument} \dots \leq \left(1 - \frac{(\varepsilon - \delta)^2}{2}\right)^n \leq e^{-n\frac{(\varepsilon - \delta)^2}{2}},
\end{aligned}$$

which completes the proof of Corollary 4. \square

In Xing [10][11] and Xing and Ranneby [12] we developed an approach to estimate the expectation in Corollary 2, where we used the Hausdorff α -entropy with the following definition.

Definition 2. Let $\alpha \geq 0$ and $\mathcal{G} \subset \mathbb{F} \subset L_\mu$. For $\delta > 0$, the Hausdorff α -entropy $J(\delta, \mathcal{G}, \alpha, \Pi, d)$ of the set \mathcal{G} relative to the prior distribution Π and the metric d is defined as

$$J(\delta, \mathcal{G}, \alpha, \Pi, d) = \log \inf \sum_{j=1}^N \Pi(B_j)^\alpha,$$

where the infimum is taken over all coverings $\{B_1, B_2, \dots, B_N\}$ of \mathcal{G} , where N may take ∞ , such that each B_j is contained in some ball $\{f : d(f, f_j) < \delta\}$ of radius δ and center at $f_j \in L_\mu$.

Note that $C(\delta, \mathcal{G}, \alpha, \Pi, d) := e^{J(\delta, \mathcal{G}, \alpha, \Pi, d)}$ is called the Hausdorff α -constant of the subset \mathcal{G} . It was proved in [12] that for any $0 \leq \alpha \leq 1$ and $\mathcal{G} \subset \mathbb{F}$,

$$C(\delta, \mathcal{G}, \alpha, \Pi, d) \leq \Pi(\mathcal{G})^\alpha N(\delta, \mathcal{G}, d)^{1-\alpha} \leq N(\delta, \mathcal{G}, d),$$

where $N(\delta, \mathcal{G}, d)$ stands for the minimal number of balls of d -radius δ needed to cover \mathcal{G} . Throughout this paper, by d_0 we denote a metric such that it is bounded above by the Hellinger metric H and $d_0(\cdot, g)^s$ is convex in \mathbb{F} for some positive constant s and any $g \in \mathbb{F}$. For such a metric d_0 we have

Lemma 1. (Xing [11]). Let $0 < \alpha \leq 1$, $\mathcal{G} \subset \mathbb{F}$ and $D_{r\varepsilon} = \{f \in \mathcal{G} : d_0(f, f_0) \geq r\varepsilon\}$ with $r > 2$ and $\varepsilon > 0$. Then we have

$$E_{f_0} \left(\int_{D_{r\varepsilon}} R_n(f) \Pi(df) \right)^\alpha \leq C(\varepsilon, D_{r\varepsilon}, \alpha, \Pi, d_0) e^{\frac{\alpha-1}{2}(r-2)^2 n \varepsilon^2}.$$

As a consequence of Corollary 2 we obtain the following result which essentially is Theorem 1 of Xing and Ranneby [12].

Corollary 5. Let $0 < \alpha < 1$ and $\beta_1 > 0$. Suppose that the true density f_0 is in the Kullback-Leibler support of Π and suppose that for any $\varepsilon > 0$ there exist positive a constant $\beta_\varepsilon > 0$ and a sequence of $\mathcal{G}_n \subset \mathbb{F}$ such that

- (a) $\Pi(\mathbb{F} \setminus \mathcal{G}_n) \leq e^{-n\beta_\varepsilon}$;
- (b) $C(\varepsilon, \mathcal{G}_n, \alpha, \Pi, d_0) \leq e^{n\beta_1\varepsilon^2}$.

Then for any $\varepsilon > 0$,

$$\Pi(f : d_0(f, f_0) \geq \varepsilon \mid X_1, X_2, \dots, X_n) \longrightarrow 0$$

almost surely as $n \rightarrow \infty$.

Note that for $\alpha = 1$ we have $C(\varepsilon, \mathcal{G}_n, \alpha, \Pi, d_0) = \Pi(\mathcal{G}_n) \leq 1$ which yields (b) of Corollary 5. Hence Corollary 5 does not hold when $\alpha = 1$.

Proof of Corollary 5. Assume that r is a large positive constant which will be determined later. For any $\varepsilon > 0$, by the inequality $(s + t)^\alpha \leq s^\alpha + t^\alpha$ for $s, t \geq 0$ we have

$$\begin{aligned} & E_{f_0} \left(\int_{\{f: d_0(f, f_0) \geq r\varepsilon\}} R_n(f) \Pi(df) \right)^\alpha \\ & \leq E_{f_0} \left(\int_{\mathbb{F} \setminus \mathcal{G}_n} R_n(f) \Pi(df) \right)^\alpha + E_{f_0} \left(\int_{\{f \in \mathcal{G}_n : d_0(f, f_0) \geq r\varepsilon\}} R_n(f) \Pi(df) \right)^\alpha, \end{aligned}$$

which, by Hölder's inequality and Lemma 1, does not exceed

$$\begin{aligned} & \left(E_{f_0} \int_{\mathbb{F} \setminus \mathcal{G}_n} R_n(f) \Pi(df) \right)^\alpha + C(\varepsilon, \mathcal{G}_n, \alpha, \Pi, d_0) e^{\frac{\alpha-1}{2}(r-2)^2 n \varepsilon^2} \\ & \leq \left(\int_{\mathbb{F} \setminus \mathcal{G}_n} E_{f_0} R_n(f) \Pi(df) \right)^\alpha + e^{\beta_1 n \varepsilon^2 + \frac{\alpha-1}{2}(r-2)^2 n \varepsilon^2} \\ & \leq e^{-n\beta_\varepsilon\alpha} + e^{\beta_1 n \varepsilon^2 + \frac{\alpha-1}{2}(r-2)^2 n \varepsilon^2}. \end{aligned}$$

Take r so large that $\beta_1 + \frac{\alpha-1}{2}(r-2)^2 < -\beta_\varepsilon$. Then we have

$$E_{f_0} \left(\int_{\{f: d_0(f, f_0) \geq r\varepsilon\}} R_n(f) \Pi(df) \right)^\alpha \leq e^{-\frac{n\beta_\varepsilon\alpha}{2}} \quad \text{for all large } n,$$

which by Corollary 2 completes the proof of Corollary 5. \square

Finally, we present a both necessary and sufficient theorem for in-probability exponential consistency of posteriors, which is an analogue of Theorem 1.

Theorem 2. Suppose that the true density f_0 is in the Kullback-Leibler support of Π and that $\{A_n\}_1^\infty$ be a sequence of subsets of \mathbb{F} . Then the following statements are equivalent.

- (i) There exists a constant $\beta_0 > 0$ such that

$$e^{n\beta_0} \Pi(A_n | X_1, X_2, \dots, X_n) \longrightarrow 0$$

in probability as $n \rightarrow \infty$.

- (ii) There exists a constant $\beta_1 > 0$ such that

$$P_{f_0}^\infty \{\Pi(A_n | X_1, X_2, \dots, X_n) > e^{-n\beta_1}\} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- (iii) There exist constants $0 < \alpha_1 \leq 1$, $\beta_2 > 0$ and a sequence $\{D_n\}_1^\infty$ of sets $D_n \subset \mathbb{X}^n$ with $P_{f_0}^\infty(D_n) \rightarrow 0$ as $n \rightarrow \infty$ such that

$$E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) \right)^{\alpha_1} \leq e^{-n\beta_2} \quad \text{for all large } n.$$

- (iv) For each $0 < \alpha \leq 1$ there exist a constant $\beta_\alpha > 0$ and a sequence $\{D_n\}_1^\infty$ of sets $D_n \subset \mathbb{X}^n$ with $P_{f_0}^\infty(D_n) \rightarrow 0$ as $n \rightarrow \infty$ such that

$$E_{f_0} \left(1_{\mathbb{X}^n \setminus D_n} \int_{A_n} R_n(f) \Pi(df) \right)^\alpha \leq e^{-n\beta_\alpha} \quad \text{for all large } n.$$

- (v) There exist a constant $\beta_3 > 0$ and a sequence $\{D_n\}_1^\infty$ of sets $D_n \subset \mathbb{X}^n$ such that $P_{f_0}^\infty(D_n) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\int_{A_n} P_f^\infty(\mathbb{X}^n \setminus D_n) \Pi(df) \leq e^{-n\beta_3} \quad \text{for all large } n.$$

The proof of Theorem 2 follows essentially from the same lines as the proof of Theorem 1 and therefore is omitted. Similar to the proof of Corollary 1, we get a result of Barron's type on in-probability posterior convergence.

Corollary 6. Suppose that the true density f_0 is in the Kullback-Leibler support of Π and that $\{A_n\}_1^\infty$ be a sequence of subsets in \mathbb{F} . Then the following statements are equivalent.

- (i) There exists a constant $\beta_0 > 0$ such that

$$e^{n\beta_0} \Pi(A_n | X_1, X_2, \dots, X_n) \longrightarrow 0$$

in probability as $n \rightarrow \infty$.

- (ii) There exist subsets V_n, W_n of \mathbb{F} , positive constants $c_1, c_2, \beta_1, \beta_2$ and a sequence of tests $\{\phi_n = \phi_n(X_1, \dots, X_n)\}$ such that

(a) $A_n \subset V_n \cup W_n$;

(b) $\Pi(W_n) \leq c_1 e^{-n\beta_1}$;

(c) $P_{f_0}^\infty \{\phi_n > 0\} \rightarrow 0$ as $n \rightarrow \infty$ and $\inf_{f \in V_n} E_f \phi_n \geq 1 - c_2 e^{-n\beta_2}$.

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